Plan:

(1) Modified propagator for photons + phonons
(2) fermion Bubble
(3) Resummation

HW 4: 8.11, 8.12, 8.13, 8.15, 8.22

Last time:

$$G_K^{\text{phonon}}(t,t') = -i\langle T\, a_K(t)\, a_K^+(t')\rangle$$

$$\tilde{G}_K^{\text{photon}}(t,t') = -i\langle T \underbrace{(a_K(t) - a_{-K}^+(t))}_{-iA(k,t)} \underbrace{(a_K^+(t') - a_{-K}(t'))}_{\overset{|||}{iA^+(k,t)}}\rangle = -i\langle T\, a_K(t)\, a_K^+(t')\rangle - i\langle T\, a_{-K}^+(t)\, a_{-K}(t')\rangle$$

$$= G_K(t,t') + G_{-K}(t',t) = -i\langle T\, A(k,t)\, A^+(k,t')\rangle = [\text{correlator of vector gauge field}]$$

Note 1: $A(k,t) = i(a_K(t) - a_{-K}^+(t)) \Rightarrow A^+(k,t) = -i[a_K^+(t) - a_{-K}(t)]$ = [matches the def above]

Note 2: In E+M all fields [e.g. E, B, A, Φ] are Real fields
There is no such thing as complex E field [Jackson uses complex numbers to simplify calculations, but in the end he takes the real part]

Hence: $E_K(t)^* = E_{-K}(t)$ and $A_K^*(t) = A_{-K}(t)$
comparing to the above, we have defined:

$$\left.\begin{aligned} A(k,t) &= i(a_K(t) - a_{-K}^+(t)) \\ A(k,t)^+ &= -i(a_K^+(t) - a_{-K}(t)) \end{aligned}\right\} \quad A(-k,t)^+ = -i(a_{-K}^+(t) - a_K(t)) = i(a_K(t) - a_{-K}^+(t)) \quad (\checkmark)$$

What is the effect of introducing $\tilde{G}^{ph}$ into our calculation?

$$(\text{bosons}) = \langle T\, (a_K(t) - a_{-K}^+(t))\, (\hat{a}_{-q_1}(t_1) - \hat{a}_{q_1}^+(t_1))(\hat{a}_{-q_2}(t_2) - \hat{a}_{q_2}^+(t_2))(a_K^+(t') - a_{-K}(t'))\rangle$$

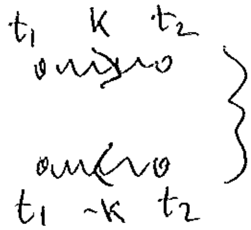
[I dropped the disconnected part]

$$= -[G_K(t,t_1) + G_{-K}(t_1,t)][G_K(t_2,t') + G_{-K}(t',t_2)] + \ldots = -\tilde{G}_K(t,t_1)\,\tilde{G}_K(t_2,t') + \ldots$$

$$\Rightarrow \text{just replace: } G_p^{ph(0)}(t_i,t_j) \longrightarrow \tilde{G}_p^{ph(0)}(t_i,t_j)$$

$\Rightarrow$ compute the fermion bubble on page 10 + 11

note 3: for photons we want to propagate the A-field ⇒ we can see this from the structure of vac. polarization, the photon Green functions always come in pairs



$$G_{k}(t_1-t_2) + G_{-k}(t_2-t_1) \equiv \tilde{G}(k,t_1-t_2)$$

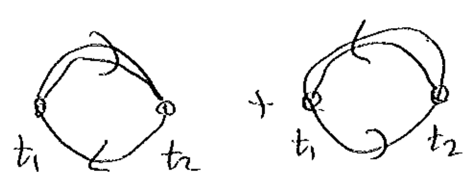
upon fourier transforming, we find

$$\tilde{G}(k,\omega) = G(k,\omega) + G(-k,-\omega) = \frac{1}{\omega-\omega_k+i\epsilon} + \frac{1}{-\omega-\omega_k+i\epsilon}$$

$$= \frac{2\omega_k}{\omega^2-\omega_k^2+i\epsilon} \qquad [\text{D.S. page 396}]$$

where $\omega_k = c|k|$ is the photon dispersion.

The Fermion Bubble:



$$\equiv \Sigma_{q_1}(t_1,t_2) = \int dp_1\, G^{(0)}_{v,p_1}(t_2-t_1)\, G^{(0)}_{c,p_1+q_1}(t_1-t_2) + G^{(0)}_{c,p_1-q_1}(t_2-t_1)\, G^{(0)}_{v,p_1}(t_1-t_2)$$

Let us fourier transform. Using the fact that the system is time translation invariant we set $t_2=0$.

$$\Sigma_{q_1}(\omega) = \int e^{-i\omega t_1}\,\Sigma(t_1)\,dt_1 = \int dt_1\, e^{-i\omega t_1}\int dp_1 \Big[G^{(0)}_{v,p_1}(-t_1)\, G^{(0)}_{c,p_1+q_1}(t_1) + G^{(0)}_{c,p_1-q_1}(-t_1)\, G^{(0)}_{v,p_1}(t_1)\Big]$$

1

$$\Sigma_q(\omega) = \int dt_1\, e^{-i\omega t_1} \int d\omega_1\, d\omega_2 \int đp_1 \Big[G^{(0)}_{v,p_1}(\omega_1) e^{-i\omega_1 t_1}\, G^{(0)}_{c,p_1+q_1}(\omega_2) e^{i\omega_2 t_1}$$
$$+ G^{(0)}_{c,p_1-q_1}(\omega_1) e^{-i\omega_1 t_1}\, G^{(0)}_{v,p_1}(\omega_2) e^{i\omega_2 t_1} \Big]$$

$$= \int d\omega_1\, đp_1 \Big[G^{(0)}_{v,p_1}(\omega_1)\, G^{(0)}_{c,p_1+q_1}(\omega_1+\omega) + G^{(0)}_{c,p_1-q_1}(\omega_1)\, G_{v,p_1}(\omega_1+\omega) \Big]$$

$$-\omega - \omega_1 + \omega_2 = 0$$

$$= \int_{-\infty}^{\infty} d\omega \int đp_1 \Big[\frac{1}{\omega_1 - \varepsilon_{v,p} - i\varepsilon}\, \frac{1}{\omega + \omega_1 - \varepsilon_{c,p+q} + i\varepsilon} + \frac{1}{\omega_1 - \varepsilon_{c,p} + i\varepsilon} \times \frac{1}{\omega_1 + \omega - \varepsilon_{v,p_1+q_1} - i\varepsilon} \Big]$$

$$\omega_1 = \varepsilon_{v,p} + i\varepsilon \qquad\qquad \omega_1 = \varepsilon_{v,p+q} - \omega$$

$$= \int đp_1 \Big[\frac{1}{\varepsilon_{v,p_1} - \varepsilon_{c,p_1+q_1} + \omega + i\varepsilon} + \frac{1}{-\omega + \varepsilon_{v,p_1+q_1} - \varepsilon_{c,p_1} + i\varepsilon} \Big]$$

$$= \int đp_1 \Big[\frac{1}{\omega - (\varepsilon_{c,p_1+q_1} - \varepsilon_{v,p_1}) + i\varepsilon} - \frac{1}{\omega - (\varepsilon_{v,p_1+q_1} - \varepsilon_{c,p_1}) - i\varepsilon} \Big]$$

$\boxed{\Rightarrow \text{assume } M \to \infty}$

$$= \int đp_1 \Big[\frac{1}{\omega - \omega_0 + i\varepsilon} - \frac{1}{\omega + \omega_0 - i\varepsilon} \Big] = [\#] \Big[\frac{2\omega_0 - i\varepsilon}{\omega^2 - \omega_0^2 + i\varepsilon} \Big]$$

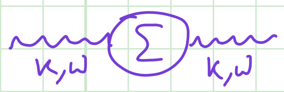
$$[\omega - (\omega_0 - i\varepsilon)][\omega + (\omega_0 - i\varepsilon)]$$

$$\omega^2 - (\omega_0 - i\varepsilon)^2 = \omega^2 - \omega_0^2 + i\varepsilon$$

note sim. to the Boson Propagator!

$$\Sigma_K(\omega) = [\#] \frac{2\omega_0}{\omega^2 - \omega_0^2 + i\varepsilon}$$

Diagrammatically we can represent the fermion bubble as:



$$= \tilde{G}_K^{ph(0)}(\omega)\, \Sigma_K(\omega)\, \tilde{G}_K^{ph(0)}(\omega)$$

photon $\to$ bubble $\to$ photon

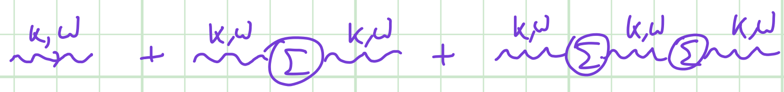
$\Sigma_K(\omega)$ is called the photon self energy

Resummation of diagrams:

We can improve on the photon propagator without much additional work using diagram resummation.

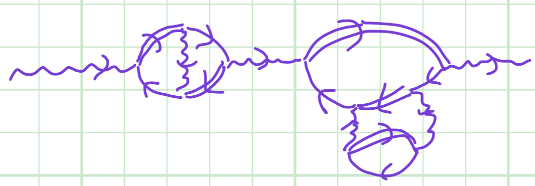
Consider the following series of diagrams that make up the photon propagator:

$\tilde{G}_K^{photon}(\omega) =$



$$= \tilde{G}_K^{ph(0)}(\omega) + \tilde{G}_K^{ph(0)}(\omega)\, \Sigma_K(\omega)\, \tilde{G}_K^{ph(0)}(\omega) + \tilde{G}_K^{ph(0)}(\omega)\, \Sigma_K(\omega)\, \tilde{G}_K^{ph(0)}(\omega)\, \Sigma_K(\omega)\, \tilde{G}_K^{ph(0)}(\omega) + \ldots$$

Note 1: This series is, to be sure, missing some terms $\Rightarrow$ examples?



etc.

Note 2: This series is easy to sum using a simple "trick"

(a) Consider the power series in small $\Sigma_K(\omega)$:

$$\frac{\tilde{G}_K^{ph(0)}(\omega)}{1 - \tilde{G}_K^{ph(0)}(\omega)\, \Sigma_K(\omega)} = \tilde{G}_K^{ph(0)}(\omega) + \left[\tilde{G}_K^{ph(0)}(\omega)\, \Sigma_K(\omega)\right] \tilde{G}_K^{ph(0)}(\omega) + \left[\tilde{G}_K^{ph(0)}(\omega)\, \Sigma_K(\omega)\right]^2 \tilde{G}_K^{ph(0)}(\omega) + \ldots$$

$$= \tilde{G}_K^{ph}(\omega)$$ [Exactly the answer that we were looking for]

(b) What is the meaning of the above expression? Multiply both sides by the denominator:

$$\left(1-\tilde{G}_K^{ph(0)}(\omega)\Sigma_K(\omega)\right)\tilde{G}_K^{ph}(\omega)=\tilde{G}_K^{ph(0)}(\omega)$$

rearrange $\Rightarrow$ $\tilde{G}_K^{ph}(\omega)=\tilde{G}_K^{ph(0)}(\omega)+\tilde{G}_K^{ph(0)}(\omega)\,\Sigma_K(\omega)\,\tilde{G}_K^{ph}(\omega)$ [Dyson eq.]

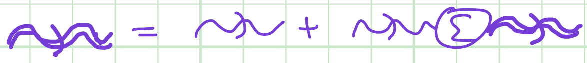
This equation constructs the full photon Green function iteratively. Let's Demonstrate diagrammatically.

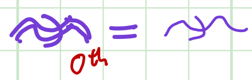
 $=\tilde{G}_K^{ph(0)}(\omega)$

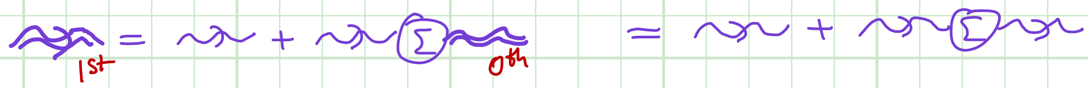
 $=\tilde{G}_K^{ph}(\omega)$

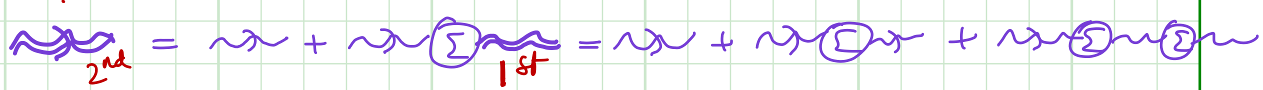
bold

using diagrams the Dyson equation looks like this:



at (0)-th order 

at (1)-st order 

at (2)-nd order 

etc ...

series generated recursively.

Note 3: How is the photon propagator affected by semiconductor?

let's plug $\tilde{G}_K^{ph(0)}(\omega)=\dfrac{2\omega_K}{\omega^2-\omega_K^2+i\varepsilon}$ into the Dyson eq:

$$\tilde{G}_K^{ph}(\omega)=\frac{\tilde{G}_K^{ph(0)}(\omega)}{1-\tilde{G}_K^{ph(0)}(\omega)\Sigma_K(\omega)}=\frac{2\omega_K}{\omega^2-\omega_K^2+i\varepsilon}\left[1-\frac{2\omega_K}{\omega^2-\omega_K^2+i\varepsilon}\Sigma_K(\omega)\right]^{-1}$$

$$=\frac{2\omega_K}{\cancel{\omega^2-\omega_K^2+i\varepsilon}}\left[\frac{\omega^2-\omega_K^2+i\varepsilon-2\omega_K\Sigma_K(\omega)}{\cancel{\omega^2-\omega_K^2+i\varepsilon}}\right]^{-1}=\frac{2\omega_K}{\omega^2-\omega_K^2-2\omega_K\Sigma_K(\omega)+i\varepsilon}$$

$\Rightarrow$ It looks like $\Sigma_K(\omega)$ is shifting the energy, ω_K, of the photon.

Hence $\Sigma_K(\omega)$ is called the self-energy.

Note 4: What is the relation between $\Sigma_K(\omega)$ and quantities we know like χ?

Conventionally in optics, if we want photon propagation in medium, we change the speed of light: $\omega_K=cK\to\tilde{\omega}_K=\tilde{c}K$. Let's compare photon propagating with alternate speed of light to the above answer:

In medium :

$$\tilde{G}^{ph}_{k}(\omega) = \frac{2\tilde{\omega}_k}{\omega^2 - \tilde{\omega}_k^2 + i\varepsilon} \Rightarrow \text{pole located at : } \omega^2 = \tilde{\omega}_k^2 = \tilde{c}^2 k^2 = \frac{c^2k^2}{n^2}$$

Our calculation, pole located at : $\omega^2 = \omega_k^2 + 2\omega_k \Sigma_k = c^2k^2 + 2ck\,\Sigma_k(\omega)$

Comparing the two : $\frac{c^2k^2}{n^2} = c^2k^2 + 2ck\,\Sigma_k(\omega) \Rightarrow \cancel{c}^2 k^{\cancel{2}}(1-n^2) = 2\cancel{ck}\,\Sigma_k(\omega)$

$$\boxed{(1-n^2) = -\chi = \frac{2\Sigma_k(\omega)}{ck}}$$

$\Rightarrow$ Next time $\rightarrow$ finite temperature diagrams

$\Rightarrow$ linear response at $T \neq 0$, e.g. conductivity